31.10.2022

4. Counting faces

MAIN QUESTIONS:

- How many δ-dimensional faces can a polytope have for some $\delta \in \{0, \dots, d-1\}$?
- What relations exist between these face numbers?
- Can we completely characterize possible face numbers of polytopes?

Def: The list of numbers

usually not included

$$\vec{f} = \vec{f}(P) = (\underbrace{f_{-1}}_{=1}, f_0, f_1, \dots, f_{d-1}, \underbrace{f_d}_{=1})$$

with $f_\delta = f_\delta(P) := \#$ faces of dimension δ is called the *f-vector* of P.

Eg.: 2D: $f_0 = f_1$ and $f_0, f_1 \geq 3$ completely characterizes f-vectors in dimension $d = 2$.

4.1. Euler's polyhedral formula and some consequences

$$V - E + F = 2$$

or $$f_0 - f_1 + f_2 = 2$$

← holds for all connected planar graphs

Ex: prove it by inductively adding vertices and edges.

- f-vectors of 3-polytopes have been completely characterized

Thm: (Steinitz)

$\vec{f} = (f_0, f_1, f_2)$ is an f-vector of a 3-polytope iff

(i) $f_0 - f_1 + f_2 = 2$ (Euler)

(ii) $4 \le f_0 \le 2f_2 - 4$

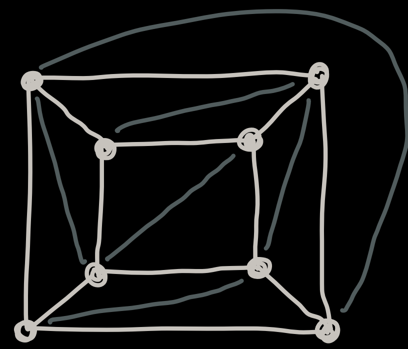
(iii) $4 \le f_2 \le 2f_0 - 4$

dual

NOTE: polar duality flips the f-vector

$$\begin{pmatrix} f_0 \\ f_1 \\ f_2 \end{pmatrix} \underset{P \qquad P^\circ}{\longleftrightarrow} \begin{pmatrix} f_2 \\ f_1 \\ f_0 \end{pmatrix}$$

- no proof, but I tell you where (ii) and (iii) come from:
  - try to maximize the number of edges and 2-faces without adding vertices by adding edges until you can't without making the graph non-planar



$\longrightarrow$ graph becomes a **triangulation**

(= every 2-face is a triangle)

- let's count incident edge-face pairs in two ways

$$2f_1 = \sum_{e} \underbrace{\sum_{f \ni e} 1}_{2} = \sum_{f} \underbrace{\sum_{e \in f} 1}_{3} = 3 f_2$$

$\longrightarrow \quad f_1 = \frac{3}{2} f_2$

$\longrightarrow \quad 2 = f_0 - f_1 + f_2 = f_0 - \frac{3}{2} f_2 + f_2$

$\quad = f_0 - \frac{1}{2} f_2$

(iii) $\longrightarrow \quad f_2 \overset{\leq}{=} 2f_0 - 4$

since we maximized f_2 this is an inequality for general planar graphs.

(ii) follows from duality

Some consequences of Euler's polyhedral formula

- let's count incident vertex-edge pairs in general planar graphs:

$$2f_1 = \sum_v \delta(v) = \bar{\delta} f_0 \quad \longrightarrow \quad f_0 = \frac{2f_1}{\bar{\delta}}$$

vertex degree average degree

- and incident face-edge pairs:

$$2f_1 = \sum_f g(f) = \bar{g} f_2 \quad \longrightarrow \quad f_2 = \frac{2f_1}{\bar{g}}$$

gonality (number of vertices of a 2-face) average gonality

- plug into Euler's polyhedral formula

$$2 = f_0 - f_1 + f_2 = \frac{2}{\bar{\delta}} f_1 - f_1 + \frac{2}{\bar{g}} f_1$$

$$\longrightarrow \boxed{\frac{1}{\bar{\delta}} + \frac{1}{\bar{g}} = \frac{1}{2} + \frac{1}{f_1} > \frac{1}{2}}$$

- we know that $\bar{\delta} \geq 3$ (Balinski)

$$\longrightarrow \frac{1}{\bar{g}} > \frac{1}{2} - \frac{1}{\bar{\delta}} \geq \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

$$\longrightarrow \bar{g} < 6$$

(likewise: $\bar{\delta} < 6$ by duality)

Cor:
- Every 3-polytope has a 2-face with at most five vertices
- Every 3-polytope has a vertex of degree at most five.

Ex: if all vertex-degrees are even then P has a triangular 2-face

≈ "Ramsey theory for polytopes"

Cor: Every polytope has a 2-face that is a triangle, quadrangle or pentagon.

Proof: every polytope has a 3-face to which the previous corollary applies. □

BUT: more is known!

Thm: (Kalai) Every d-polytope with $d \geq 5$ has a 2-face that is either a triangle or quadrangle.

Note: the 4-dimensional 120-cell has only pentagonal 2-faces.

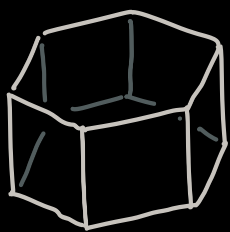
Can similar things be said about higher-dimensional faces?

OPEN: Do all sufficiently high-dimensional polytopes have a 3-face that is either a tetrahedron or a cube?

(not included in the lecture)

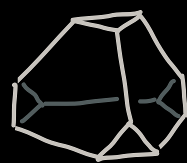
Interlude: is a polytope determined by its f-vector?

- No, already not true in dimension 3.



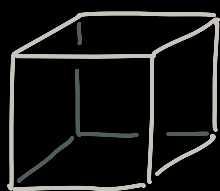
6-gonal prism

and



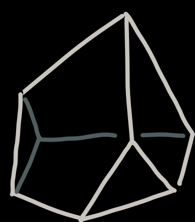
truncated tetrahedron

have the same f-vector $(12, 24, 8)$



cube

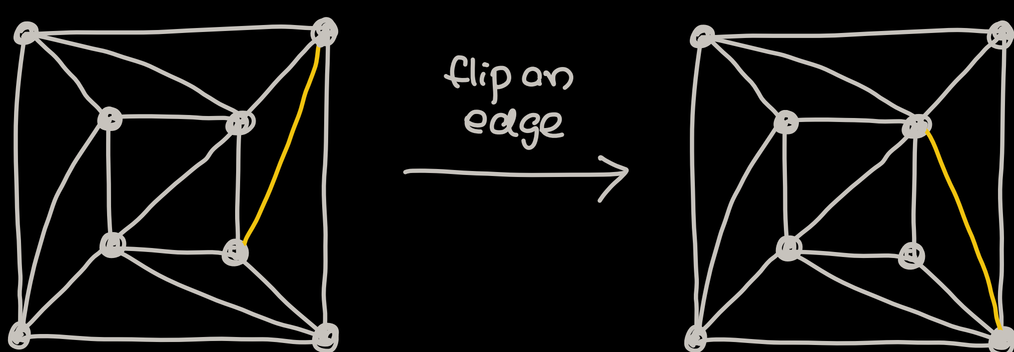
and



pentagonal wedge

have the same f-vector $(8, 12, 6)$

- Simple procedure for creating counterexamples:



A glimpse at the Euler-Poincaré identity

Q: Does Euler's polyhedral formula generalize to higher dimensions? And in which form?

2D	$f_0 - f_1$	$=$	0
3D	$f_0 - f_1 + f_2$	$=$	2
4D	$f_0 - f_1 + f_2 - f_3$	$=$	0
5D	$f_0 - f_1 + f_2 - f_3 + f_4$	$=$	2
$\vdots$			$\vdots$

Why this pattern?

The pattern becomes clearer if we include f_{-1} and f_d

3D $\quad -f_{-1} + f_0 - f_1 + f_2 - f_3 = 0$

$$\rightarrow \quad f_0 - f_1 + f_2 = \underbrace{f_{-1}}_{=1} + \underbrace{f_3}_{=1} = 2 \quad \checkmark$$

4D $\quad -f_{-1} + f_0 - f_1 + f_2 - f_3 + f_4 = 0$

$$\rightarrow \quad f_0 - f_1 + f_2 - f_3 = \underbrace{f_{-1}}_{=1} - \underbrace{f_4}_{=1} = 0 \quad \checkmark$$

The right generalization seems to be

Thm: (Euler-Poincaré identity) For a d-polytope holds

$$\sum_{i=-1}^{d} (-1)^i f_i = 0 \quad \text{or equivalently} \quad \sum_{i=0}^{d-1} (-1)^i f_i = 1-(-1)^d$$

(proof will be given next week; we need: shellability)

- This is the only linear relation that holds between the face-numbers of a general polytope!

4.2. The Dehn-Sommerville equations

- Much more can be said about the f-vector if the polytope is simple or simplicial

edge-graph is d-regular $\xleftrightarrow[\text{have a flipped f-vector}]{\text{dual}}$ all faces are simplices

Thm: (Dehn-Sommerville equations)

$P \subset \mathbb{R}^d$ simple (analogously for simplicial)

For all $k \in \{0, \ldots, d\}$ holds

$$\sum_{i=k}^{d} (-1)^i \binom{i}{k} f_i = \sum_{i=d-k}^{d} (-1)^{d-i} \binom{i}{d-k} f_i$$

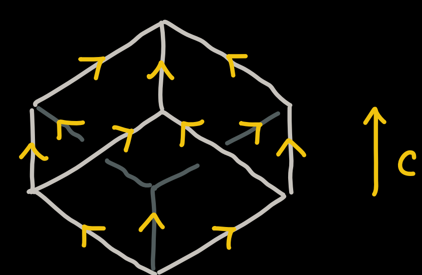
Remarks:

- might look scary but become much nicer after a basis change (see proof)
- DSeq for k and $d-k$ are equivalent
- these are $\lfloor d/2 \rfloor$ linearly independent equations

  $\longrightarrow$ knowing only half the f-vector we can reconstruct the missing entries

Proof:

- we choose a generic direction $c \in \mathbb{R}^d$ and orient the edge-graph accordingly



- define $h_i^c :=$ # vertices with out-degree $= i$

  out$(v) :=$ # edges pointing out of v

  → we show that these numbers are independent of c !

  → they are combinatorial invariants of P.

- we express the f-vector in terms of the h_i^c.

$$(*) \quad f_k = \sum_{\substack{v \in \mathcal{F}_0(P) \\ \mathrm{out}(v) \geq k}} \binom{\mathrm{out}(v)}{k} = \sum_{i=k}^{d} \binom{i}{k} h_i^c$$

Ex
1. every face of P contains a unique minimal vertex wrt. the orientation
2. every set of k edges emanating from a vertex belongs to a unique face (only true for simple polytopes)

- we show that therefore the h_i^c are also determined by the f-vector, independent of c.

- define $\mathcal{F}(t) := \sum_{i=0}^{d} f_i t^i$ and $\mathcal{H}^c(t) := \sum_{i=0}^{d} h_i^c t^i$

$$\longrightarrow \mathcal{H}^c(t+1) = \sum_{i=0}^{d} h_i (t+1)^i = \sum_{i=0}^{d} h_i \sum_{k=0}^{i} \binom{i}{k} t^k$$

$$= \sum_{k=0}^{d} t^k \sum_{i=k}^{d} \binom{i}{k} h_i$$

$$\overset{(*)}{=} \sum_{k=0}^{d} t^k f_k = \mathcal{F}(t)$$

$\rightarrow$ $\mathcal{H}^c$ and the h_i^c independent of c and we can write $\mathcal{H}$ and h_i instead.

Def: h-vector $\vec{h} := (h_0, \dots, h_d)$

- super important in modern polytope theory
- the "right way" to look at face numbers
- appears everywhere: - reconstruction from the edge-graph
  - computing volumes (Ehrhart theory)
  - shellings ...

- since the h_i are independent of the direction vector

$$h_i = h_i^c = h_i^{-c} = h_{d-i}^c = h_{d-i}$$

1. if i edges are going out of v then $d-i$ edges are coming in.

2. $c \mapsto -c$ flips in- and out-going edges

$\rightarrow$ $\boxed{h_i = h_{d-i}}$ Dehn-Sommerville equations in h-basis.

- note that $\mathcal{H}(t) = \mathcal{F}(t-1)$, therefore

$$h_k = \sum_{i=k}^{d} (-1)^i \binom{i}{k} f_i$$

which yields the equations in f-basis. $\square$

Remarks:

- $k=0$ gives the Euler-Poincaré identity for simple / simplicial polytopes
- the h_i count something, therefore $h_i \geq 0$ and we get potentially useful inequalities:

$$h_0 = 1 \geq 0 \quad \text{trivially true}$$
$$h_1 = f_{d-1} - d \geq 0 \quad \text{easily verified}$$
$$h_2 = f_{d-2} - (d-1) f_{d-1} + \binom{d}{2} \geq 0 \quad \dots \text{not so trivial anymore}$$

- much stronger inequalities are known

Thm: the h-vector is unimodal
i.e. its entries increase, and then decrease with only one peak.

$$h_0 \leq h_1 \leq \dots \leq h_{\lfloor d/2 \rfloor} = h_{\lceil d/2 \rceil} \geq \dots \geq h_d$$

This is a deep theorem (see g-theorem later)

- If a polytope is $> \lfloor d/2 \rfloor$-neighborly then more than half of the entries of f-vector agree with simplex
  - $\longrightarrow$ f-vector is the same as simplex
  - $\longrightarrow$ **Ex:** it is a simplex

- we can compute full f-vector of cyclic polytope $C_d(n)$
  - $k \leq \lfloor d/2 \rfloor$: $\quad f_k = \binom{n}{k+1}$ (not included in the lecture)

- $k > \lfloor d/2 \rfloor$:

$$f_k = \frac{n - \delta(n-k-2)}{n-k-1} \sum_{j=0}^{\lfloor d/2 \rfloor} \binom{n-1-j}{k+1-j} \binom{n-k-1}{2j-k-1+\delta}$$

where $\delta = 0$ if d is even, and $\delta = 1$ otherwise.

$\longrightarrow$ $f_{d-1}(C_d(n)) \in O(n^{d/2})$

- The Dehn–Sommerville equations are the only linear relations between the face numbers of simple / simplicial polytopes

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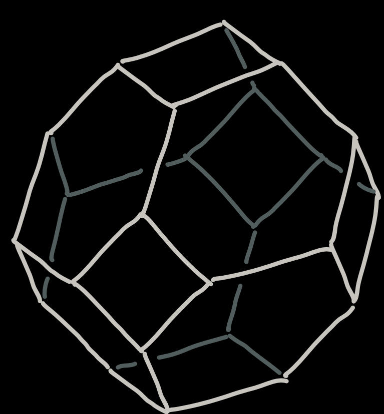
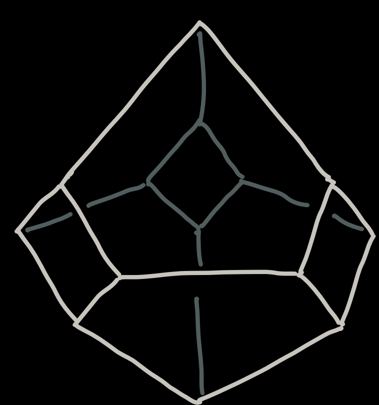
Applying the Dehn-Sommerville equations to particular polytopes for which is known that the face-numbers encode important combinatorial sequences can yield noval relations! (not included in the lecture)

Example



permutahedron
($n!$)



associahedron
(Catalan numbers)

4.3. The upper bound theorem & g-theorem

- given a d-polytope with n vertices, how many k-faces can it have?

- not only is this upper bound known, there even exists a polytope which attains this bound for all k !!

$\longrightarrow$ the cyclic polytope $C_d(n)$

Thm: (upper bound theorem)
If P is a d-polytope with n vertices then

$$f_k(P) \leq f_k\left(C_d(n)\right) \quad \text{for all } k$$

- This is one of the big theorems. Combinatorial proofs are known, but the modern approaches are via commutative algebra.

- f-vectors of simple/simplicial polytopes are completely characterized

$$f\text{-vector} \longmapsto h\text{-vector} \longmapsto g\text{-vector}$$

$$g_0 := 1, \quad g_k := h_k - h_{k-1}$$

(Actually: the g-vectors got classified)

Thm: An f-vector belongs to a simplicial polytope iff its g-vector is an M-sequence.

M-sequence means $g_0 = 1$ and

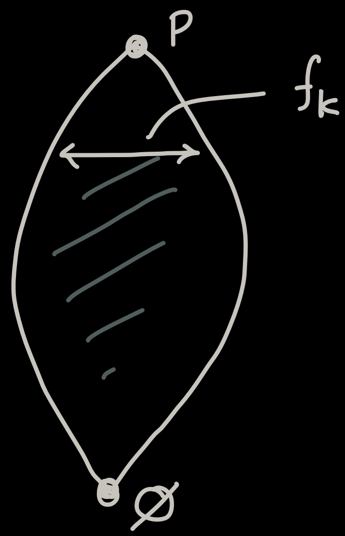
$$g_{k-1} \geq \partial^k(g_k) \geq 0$$

where $\partial^k : \mathbb{N} \longrightarrow \mathbb{N}$ is some not too complicated function.

- Only proven 2018 by Karim Adiprasito using some quite advanced algebraic techniques.

4.4. Other facts about face numbers

- is this picture of the face lattice accurate ?

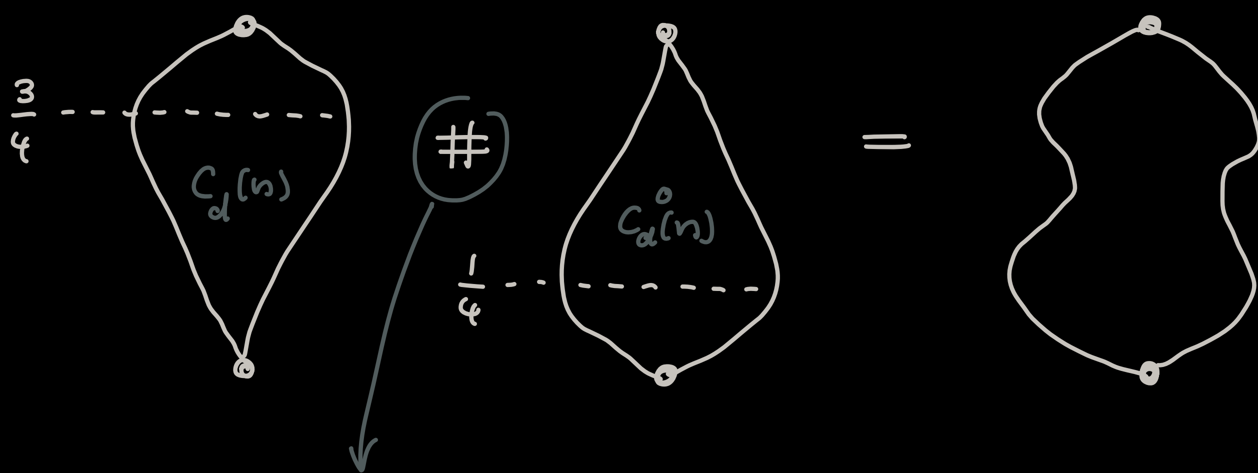


Q: is the f-vector unimodal ?

NO and for general polytopes counterexamples can be constructed "quite easily."

Example :

- f-vector of cyclic polytope $C_d(n)$ peaks at $\sim \frac{3}{4} d$ $\longrightarrow$ dual peaks at $\sim \frac{1}{4} d$



connected sum (gluing polytopes along faces) adds up the f-vectors (almost).

$\longrightarrow$ yields counterexamples with $d = 8$ and $f_0 \sim 7000$
or $d = 9$ and $f_0 \sim 1500$

- $C_d(n) \# C_d^{\circ}(n)$ is neither simple nor simplicial
- in fact, it is known that for $d \leq 15$ simple / simplicial polytopes have a unimodal f-vector.

- BUT: using the g-theorem counterexamples can be found with $d = 20$, $f_0 = 169$
  or $d = 30$, $f_0 = 47$

- Q: is it true that the face lattice is thinnest on top or bottom?

$$f_k \overset{?}{\leq} \min(f_0, f_{d-1}) \quad \text{(Bárány's conjecture)}$$

$\longrightarrow$ just solved this year (2022)! by Joshua Hinman

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- shape of 4-dimensional f-vector can be measured more precisely

$$\text{fatness}(P) := \frac{f_1 + f_2}{f_0 + f_3} \quad \begin{matrix} \leftarrow \text{thickness "in the middle"} \\ \leftarrow \text{thickness "at top and bottom"} \end{matrix}$$

OPEN: Is the fatness of 4-polytopes bounded?

- Highest known fatness is $5 + \varepsilon$

OPEN: Does there exist a 4-polytope all whose 3-faces are icosahedra?

$\longrightarrow$ if yes, this polytope would have an outstandingly large fatness (= 10?)

- A lot of ongoing research tries to generalize neighborlyness, Dehn-Sommerville, upper bounds and characterizations to other classes:
  - centrally symmetric
  - cubical (all faces are cubes)